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The cosmological problem as initial value problem on the observer's past light cone: geometry

G Dautcourt

Zentralinstitut für Astrophysik der Akademie der Wissenschaften der DDR, 1502 Potsdam-Babelsberg, R-Luxemburg Strasse 17a, German Democratic Republic

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Abstract. In the context of general relativity an approach to cosmology is suggested which claims to rest exclusively on quantities accessible (at least in principle) to the astronomical observer. No cosmological principle will be adopted as an *a priori* assumption. Conclusions leading to a certain cosmological model, or to a class of such models, will be based on data which, in principle, are observable.

The approach is based on the characteristic initial value problem for the Einstein field equations imposed on the observer's past light cone. The inner geometry of the past light cone enters the relations between observable quantities. Thus, it should become possible to determine both the cone geometry, as well as the distribution of cosmic matter near the cone, without adopting arbitrary assumptions.

The propagation equations determine the gravitational field and the matter distribution at finite distances from the past cone. In this way, cosmological principles like the Copernican principle or homogeneity requirements may become, to some degree, accessible to observational tests, provided general relativity is adopted as the correct theory of gravity.

These assertions refer to the inner domain of the past light cone, that is, to the relativistic past of the observer. Fields or matter outside the past cone are not determined without additional assumptions. Thus one achieves some separation between observationally founded statements and arbitrary assumptions in relativistic cosmology.

In the present article the first steps of the programme are carried out. The cone initial data are specified for a general dust model.

An explicit procedure in terms of successive integrations is given to determine the gravitational field near the past light cone.

1. Introduction

Facing the diversity of known cosmological models within Einstein's general theory of relativity, it is still an open question which of them corresponds to reality. Are observations able to decide this question uniquely, or is our adoption of a small class of models, such as the Friedman models, merely a reflection of our own symmetry requirements for the universe? More precisely, which part of the large-scale structure of the universe can be determined using observational data (and the powerful machinery of general relativity) alone, adopting no cosmological principle whatsoever?

This question and the way to a practicable answer within the framework of general relativity is discussed in this and further articles. In general terms the answer follows from the principles of relativistic causality.

Most cosmological experience is taken from data on the past light cone of the observer. From an observational or kinematical point of view the past light cone is the primary object in cosmology. Thus, for instance, the usual cosmological tests are tests of the structure of the nearby light cone and of the matter distribution near the cone rather than tests of global world models. However, also from the dynamical point of view, the light cone is a basic object. This relies on the fact that classical fields are described by systems of hyperbolic differential equations, for which the initial value problem on a characteristic hypersurface such as a cone is a well posed problem. It had already been suggested by Heckmann and Schucking at the 1958 Solvay congress in Brussels that the cosmological problem should be considered as a characteristic initial-value problem on the past light cone of the present observer. The peculiar property of this type of initial value problem is that with data given on the past cone, the field inside the cone can be determined, while the data do not fix the field outside the cone. One may turn this into the statement that one cannot predict the future in cosmology. For example, incoming gravitational waves of unknown intensity crossing the observer's world line in the future cannot be excluded.

Obviously, relativistic causality confines a strictly observationally based cosmology to the interior of the present light cone†. The interior need not enclose all world lines of matter, if particle horizons exist. Any statement concerned with world points outside the observable region involves assumptions which cannot be confirmed by observations made by a present time observer. The characteristic initial value problem seems to be the proper tool to separate conclusions derived from observational data alone from those involving hypotheses such as a universal, or even only a local, validity of the cosmological principle. In particular, the question of existence of a past singularity may be answered in terms of past cone data alone, without reference to the hypothetical existence of a global Cauchy surface with prescribed properties (Hawking and Ellis 1968).

Clearly, not all cosmologically important information is confined to the past cone. Incoming cosmic ray particles with non-zero rest mass are important, the high-energy tail of cosmic ray particles may be of considerable cosmological significance as well as geological data, the 'traces' of events along and around the observer's past world line. In particular, the age of the stars in the oldest globular and galactic clusters gives some important limits to cosmological parameters.

Even more important is the primordial synthesis of light elements. This process offers the possibility to restrict both the space–time geometry and the state of matter in the neighbourhood of a past section of the observer's world line, deep inside his past light cone. From the point of view of the cone initial-value problem, data of this type should be redundant because one should be able to predict them from initial data on the past cone. In practice, however, they are extremely valuable, since cosmological observations are too few in number and too closely connected with the intrinsic evolution of cosmic objects to determine all initial data with the required accuracy.

The characteristic initial-value problem was investigated several years ago for gravitational and electromagnetic, as well as spinor, fields in connection with the theory of gravitational radiation (see e.g. Sachs 1962, Penrose 1963, Dautcourt 1963).

† It is for this reason that we use the more precise term 'metagalaxy' instead of 'universe', if we refer to directly observable (or more precisely, presently observable) properties of matter and space–time on a large scale.

In the case of non-analytic fields the problem was treated by Müller zum Hagen and Seifert (1977), based on methods devloped in connection with the Cauchy problem by Choquet-Bruhat (1962, 1971) and Choquet-Bruhat and Geroch (1969) and others. An application of the characteristic initial-value problem to cosmology requires one to proceed along the following lines.

(1) One needs a description of the intrinsic geometry of a general-relativistic light cone $C^{-}(P)$. For a first discussion of the 'nearby light cone' and its interior (which already comprehends the whole observable metagalaxy) this description need only extend to the first singularity (caustic) on the past cone.

(2) The type and number of required initial data for the space–time geometry, as well as the matter distribution, must be specified.

(3) As a first step in solving the initial value problem, one should be able to determine both the local gravitational field and the motion of matter on $C^{-}(P)$. This requires one to calculate the full space–time Riemann tensor as well as, e.g., the kinematical invariants expansion, shear and rotation of the matter congruence on the past light cone.

(4) The relations between observable quantities and quantities describing the light cone structure as well as the matter distribution at the cone must be derived. However, in some cases the relations between observable cosmological quantities contain not only the inner geometry of the past light cone, but some other geometrical quantities which can be expressed in terms of the characteristic initial data.

(5) These theoretical relations following from general relativity should be compared with the corresponding empirical ones in order to determine the intrinsic cone structure and matter distribution at the cone or, more realistically, to put limits on them.

(6) One then has to determine both the gravitational field and the matter distribution inside the past cone by using the data at the cone surface. Notice that this might not give a unique answer since the cone initial data are only partly known from observations. Furthermore, this step requires handling of the coordinate singularities which necessarily arise inside the past light cone.

(7) A further task, among others, would be to determine from the geometry and matter distribution within the cone the frozen primordial concentraion of light elements such as hydrogen, helium and deuterium, including its possible space variation around the observer's past world line. In general, this requires numerical calculation, as do some earlier steps.

(8) A full treatment needs an extension of the initial value problem beyond the first caustic at the cone surface (this can still be done by means of local differential geometric methods) and a discussion of the global initial-value problem. In this article we begin the discussion by considering steps (1)–(3) (step (1) is discussed in §§ 2–4, steps (2) and (3) in §§ 6 and 7).

To simplify the treatment, incoherent matter is assumed. This confines the discussion to stages of the past history where the energy–stress tensor of radiation fields is negligible compared with other contributions to the right-hand side of the Einstein field equations.

Thus, our approach covers space–time regions including the quasars (if at cosmological distances) but excludes th 3 K microwave background radiation. To treat the latter, a characteristic initial-value problem including the equation of radiative transfer should be discussed.

The treatment given so far in steps (1)–(3) can be considered as the first steps of an extension of the Kristian–Sachs (1966) approach from a neighbourhood of the observer to a neighbourhood of the observer's past light cone. We adopt general relativity (we could have taken any other metric theory of gravitation) to restrict the vast possibilities of a pure cosmographic treatment of cosmology.

If this is done, a surprisingly small number of past cone initial data is needed. Part of the intrinsic cone geometry (its 'conformal structure', given by, e.g., the complex Penrose function $C_{\mu\rho\nu\sigma}p^\mu p^\nu \bar{t}^\rho \bar{t}^\sigma$) as well as—for incoherent matter—the matter density and three velocity components (six functions altogether) must be specified on the past light cone, in order to determine a general cosmological model within its observationally accessible range.

2. Observer based coordinates

The cosmological coordinates used in this article are defined entirely in terms of an observer at the world point P and his past light cone (figure 1). Let the equation of the past cone be given by $v(x^\mu)=0$ in any coordinate system x^μ with the tangential vector $p_\mu = v_{,\mu}$ being a null vector. The geodesics forming the past cone may be described by $x^\mu = x^\mu(u, w^A)$, where w^A, $A=2, 3$ are two 'transversal' parameters which are constant along a geodesic. They may serve to number the geodesics constituting the past cone. u is assumed to be an affine parameter. $p^\mu = \mathrm{d}x^\mu/\mathrm{d}u$ satisfies

$$\mathrm{d}p^\mu/\mathrm{d}u + \Gamma^\mu_{\rho\sigma}p^\rho p^\sigma = 0 \tag{2.1}$$

besides $p^\mu p_\mu = 0$. The affine parameter u is defined up to a change $u' = au + b$, where a, b possibly depend on w^A. Under this change $p^\mu \to p^{\mu'}/a$. The observer's four-velocity V^μ at P determines uniquely a preferred affine parameter u' along the past cone light rays by requiring the conditions $V^\mu p_\mu = -1/\sqrt{2}$ and $u'=0$ at the vertex P.

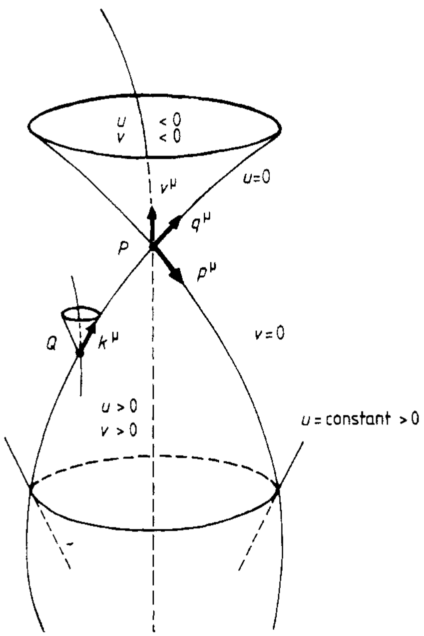

Figure 1. Schematic drawing of a general-relativistic light cone with vertex at P.

(In the following, primes will be omitted since we deal exclusively with preferred affine parameters.) The equation $u=\text{constant}$ represents regular closed two-dimensional spacelike surfaces on the past cone. One of their light normals is the tangential vector p^μ, the other (say q^μ) points out of the past cone and defines—if extended geodetically—a set $u=\text{constant}$ of cone-shaped null hypersurfaces. Each member of this set is said to be conjugate to the past cone. The set includes for, say, $u=0$ the future light cone of the observer at P. Since the observer's velocity defines also a preferred affine parameter, say v, on the future light cone, a similar set of conjugate null hypersurfaces $v=\text{constant}$ emanating from the two-dimensional surfaces $v=\text{constant}$ on the future cone can be constructed, including the past cone for $v=0$. Both sets of null hypersurfaces intersect each other in two-dimensional closed space-like surfaces, specified by given values of u and v. Two transversal coordinates w^A, $A=2, 3$, are defined as being constant along all generators (light rays) of the null hypersurfaces $v=\text{constant}$ and also constant along the rays of the past cone emanating from P. However, we may not use these parameters to number the light rays on the cone-like hypersurfaces $u=\text{constant}$ intersecting the past cone, since the w^A will change, in general, along these rays.

Geometrical constructions of this type define a null coordinate system. In terms of an arbitrary coordinate system x^μ, null coordinates are determined by the following differential equations and initial conditions. Everywhere

$$u_{,\rho}u_{,\sigma}g^{\rho\sigma}=0, \qquad v_{,\rho}v_{,\sigma}g^{\rho\sigma}=0, \qquad w^A_{,\rho}u_{,\sigma}g^{\rho\sigma}=0; \tag{2.2a,b,c}$$

on the past and future cones $u=v=0$:

$$u_{,\rho}v_{,\sigma}g^{\rho\sigma}=-1; \tag{2.3}$$

on the past cone $v=0$:

$$w^A_{,\rho}v_{,\sigma}g^{\rho\sigma}=0 \tag{2.4}$$

and at the vertex P:

$$w^A_{,\rho}V^\rho=0, \qquad u_{,\rho}V^\rho=-1/\sqrt{2}, \qquad v_{,\rho}V^\rho=-1/\sqrt{2}. \tag{2.5}$$

We have chosen the null coordinates u and v so that they are positive within the past light cone (cf figure 1). Both u and v increase into the past with $u=0$ at the vertex and $v=0$ on the past light cone. This convention is convenient for the past light cone initial-value problem. The signs and numerical values in (2.5) follow from V^μ being a time-like unit vector directed into the future. A discussion of the Cauchy problem for the system (2.2)–(2.5) shows that u, v and x^A are in fact uniquely fixed apart from a transformation of the type $w^{A'}=w^{A'}(w^A)$. Written in terms of null coordinates the metric is given by

$$ds^2=m^2\,du^2+2h\,du\,dv+2k_A\,dw^A\,du+g_{AB}\,dw^A\,dw^B. \tag{2.6}$$

The null coordinates are numbered according to

$$x^0=u, \qquad x^1=v, \qquad x^A=w^A. \tag{2.7}$$

The covariant and contravariant components are related through

$$\begin{aligned} &g^{00}=0, \qquad g^{01}=1/h, \qquad g^{0A}=0, \qquad g^{11}=0\\ &g^{1A}=-g^{AB}k_B/h, \qquad g^{AC}g_{BC}=\delta^A_B, \qquad m^2=k_Ak_Bg^{AB}. \end{aligned} \tag{2.8}$$

The signature is chosen as $(+++-)$, so g_{AB} represents a positive-definite metric. The conditions (2.3)–(2.5), furthermore, require

$$\begin{aligned} &k_A = 0 && \text{at } v = 0 \text{ (past cone)}, \\ &h = -1 && \text{at } v = 0 \text{ and } u = 0 \text{ (past and future cone)}, \\ &V^1 = V^0 = -1/\sqrt{2} && \text{at } P. \end{aligned} \tag{2.9}$$

For the Riemann and Ricci tensors in the null coordinate system see the appendix and § 6.

We note some properties of the coordinates u, v, w^A. In general u and v do not represent affine parameters along the relevant rays, with the exception of the parameters along the rays on the past and future light cone of P. This is seen from the equation for the null geodesics, which has its standard form (1) only on the (full) light cone through P. (One easily verifies that (1) is automatically satisfied on the past cone, if the coordinate system u, v, w^A is introduced. The same is true for the corresponding equation for the future cone generators.) Also, as noted above, neither will w^A be constant along the rays of both sets of null hypersurfaces u = constant and v = constant. The description of the space–time metric in terms of the coordinates u, v and w^A is free of coordinate singularities only in domains without intersection of light rays. Caustics consisting of points where light rays with infinitesimally differing transversal parameters w^A intersect will inevitably occur, however. Note first, that the point P is itself a degenerate focal point (vertex). Secondly, the rays on the hypersurfaces v = constant <0 (figure 1) intersect each other along two-dimensional caustics, forming a time-like cone-shaped hypersurface with the vertex at P that opens into the future.

A similar time-like cone opens into the past and consists of the set of focal two-surfaces of the null hypersurfaces u = constant < 0. Thirdly, as is well known, the past light cone from P will also exhibit caustics in the past if the matter density does not decrease too fast along a null geodesic.

For the particular case of the Robertson–Walker models these caustics collapse into a second singular vertex of the past cone from P, the vertex being singular in this particular case since it coincides with cosmological singularity. If space–time is flat, all null hypersurfaces constructed so far turn out to be cones, the time-like singular cones collapse onto a straight time-like line and there are no caustics on the past light cone of P. Although more complicated, the additional singularities present in a curved space–time endowed with null coordinates have essentially the same nature as the singularities arising from the use of spherical or cylindrical coordinates at the centre or on the axis. We denote that part of the past light cone through P which extends to the first caustic by $C^-(P)$ or C^-. Similarly $C^+(P)$ or C^+ denotes the corresponding part of the future cone. The interior of the past light cone which can be reached by past directed null geodesics starting from $C^-(P)$ will be denoted by $D^-(P)$ (and similarly $D^+(P)$ for the inner domain of the future cone).

The coordinate system employed here is almost completely determined by the observer. The only freedom still left is a change $w^A \to w^{A'}(w^A)$ of the transversal parameters w^A for the past cone, that is an arbitrary change of the two coordinates which the observer uses to describe positions on his sky. We shall assume the observer to be a privileged or ideal observer, that is we suppose that there is neither relative motion nor acceleration of the observer relative to the mean motion of cosmic matter in his surroundings. Effects of this kind, for instance the solar system motion in the galaxy, or the motion of the local group within the Virgo supercluster are easily accounted for. But even for an 'ideal' observer, the coordinate system reflects the particular position of the observer in the world and not necessarily the geometrical structure of space–time. On the other hand, for the interpretation of cosmological observations without *a priori* assumptions for the large-scale geometry, this coordinate system is most convenient.

Of course, we could have introduced some other type of null coordinate system, for example a coordinate system based on the past light cones $v^* = \text{constant}$ with vertices on the observer's world line L. A space–time point Q may be described here by the proper time τ ($v^* = \tau$ at L) of the observer, by an affine parameter u^* on the cones $v^* = \text{constant}$ and by the sky direction of the rays in the observer's rest frame. This system avoids the time-like cones of focal points, but it is not directly accessible to the observer. Ellis (1980), in an approach to cosmology very similar to that given here, has used this coordinate system, with the line element written in the form $ds^2 = a\,dw^2 + 2b\,dw\,dy + 2v_A\,dx^A + h_{AB}\,dx^A\,dx^B$ (y as an affine parameter on the past light cone and $w = \tau$). He noted that a, b and v_A are not directly measurable by past light cone observations of the observer at P. Notice that the metric components are still subject to coordinate gauges. The values of the quantities a, b, v_A on the past light cone are spurious indeed, as may be seen by a consideration similar to that given in this section.

3. The gravitational field around the vertex

At the vertex P the null coordinate system $x^\mu = (u, v, w^A)$ becomes singular. The behaviour of the metric $g_{\mu\nu}$ near P is derived from the relations

$$g_{\mu\nu} = (\partial\bar{x}^\alpha/\partial x^\mu)(\partial\bar{x}^\beta/\partial x^\nu)\bar{g}_{\alpha\beta}, \tag{3.1}$$

where $\bar{x}^\alpha$ is a regular coordinate system at P. The functions $\bar{x}^\alpha(x^\mu)$ exist even at the vertex, but not the inverse functions $x^\mu = x^\mu(\bar{x}^\alpha)$. For simplicity, we choose, as did Kristian and Sachs (1966), a Riemannian normal coordinate system $\bar{x}^\alpha$ at P. Besides $\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ at P, all geodesics from P have the equation

$$\frac{d^2\bar{x}^\mu}{ds^2} = 0 \qquad \text{or } \Gamma^\mu_{\rho\sigma}(\bar{x}^\alpha)\frac{d\bar{x}^\rho}{ds}\frac{d\bar{x}^\sigma}{ds} = 0 \tag{3.2}$$

(s is the arc length or an affine parameter in the case of null geodesics). Near P the metric tensor is represented by the power series

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \tfrac{1}{6}(\bar{R}_{\mu\rho\nu\sigma} + \bar{R}_{\mu\sigma\nu\rho})\bar{x}^\rho\bar{x}^\sigma, \ldots, \tag{3.3}$$

where $\bar{x}^\rho$ is the coordinate distance to P, and the Riemann tensor is taken at P (the $\bar{x}^\rho$ are still subject to Lorentz transformations).

Expanding the functions $\bar{x}^\alpha$ into powers of u and v, and substituting this power series into (3.1), with $\bar{g}_{\mu\nu}$ from (3.3), a number of relations between the power series coefficients of $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ are obtained. Some of them already follow from (3.2). One derives

$$\bar{x}^\mu = \bar{q}^\mu v + \bar{p}^\mu u + \bar{a}^\mu uv^2 + \bar{b}^\mu u^2 v + O(u^i v^k) \tag{3.4}$$

($i + k \geqslant 4$) for the connection between null and normal coordinates. Here $\bar{p}^\mu$ and $\bar{q}^\mu$ (equal to the limiting values of $\bar{p}^\mu = \bar{g}^{\mu\nu}v_{,\nu}$ and $\bar{q}^\mu = \bar{g}^{\mu\nu}u_{,\nu}$) span the null direction in the vertex, and also $\bar{a}^\mu$ and $\bar{b}^\mu$ depend only on the transversal coordinates w^A. The components of $g_{\mu\nu}$ near P can then be represented as a power series in u and v ($o(u^4)$ stands for $o(u^i v^k)$ with $i+k=4$):

$$m = o(u^4), \qquad h = -1 + h^{(11)}uv + o(u^3),$$
$$k_A = uv^2 k_A^{(12)} + u^2 v k_A^{(21)} + v^3 k_A^{(03)} + o(u^4). \tag{3.5}$$
$$g_{AB} = (u-v)^2 \underset{2}{g}_{AB}/2 + o(u^4)$$

The coefficients of the power series depend linearly on the Riemann tensor $\bar{R}_{\mu\nu\rho\sigma}$ in P, except for the first term in g_{AB}, which can be given a standard form using a suitable transformation $w^{A'} = w^{A'}(w^A)$ (the Riemann tensor occurs in the neglected terms).

Thus the components of the metric tensor in the null coordinate system have a definite limiting behaviour near the vertex. Apart from h, which tends to -1, all other components of $g_{\mu\nu}$ (as well as the determinant $|g_{\mu\nu}|$) vanish if P is approached. This is typical for a coordinate singularity caused by a missing one-to-one map of coordinates and world points (P is presented by $u = v = 0$ and arbitrary w^A). We note briefly that an expansion like (3.5) also allows one to study the coordinate singularities on the null hypersurfaces $v = \text{constant} > 0$ for small $u > 0$. The hypersurfaces $v = \text{constant}$ do not in general represent cones. The vertex is replaced by caustic surfaces with equation determined from $|g_{AB}| = 0$.

4. Light cone geometry

For the past directed null geodesics through P we have from (3.2) in the Riemannian coordinate system

$$\bar{x}^\mu = \bar{p}^\mu(w^A)u, \tag{4.1}$$

where the two parameters w^A span all null directions at P. In fact, (4.1) is the equation of the light cone through P. The induced inner cone metric is given by ($w^1 = u$, $i = 1, 2, 3$)

$$g_{ik} = (\partial \bar{x}^\mu/\partial w^i)(\partial \bar{x}^\nu/\partial w^k)\bar{g}_{\mu\nu}. \tag{4.2}$$

From (4.1), $g_{i1} = 0$: g_{ik} is degenerate with matrix rank two. The basic quantities for the inner cone geometry are, therefore, the transversal components g_{AB} ($A, B = 2, 3$). g_{AB} can be represented by a complex two-dimensional null vector t_A with $t^A t_A = 0$, $t^A t_A^* = 1$ (indices are moved with g_{AB}):

$$g_{AB} = t_A t_B^* + t_A^* t_B. \tag{4.3}$$

t_A is fixed up to a rotation $t_A \to t_A \exp(i\phi)$. Rotation coefficients with respect to t_A are introduced by

$$\rho = -t^A \dot{t}_A^*, \qquad \sigma = -t^{A*}\dot{t}_A^*, \qquad \tau = t^{A*}t^B(t_{A,B}^* - t_{B,A}^*) \tag{4.4}$$

(the point denotes $\partial/\partial u$), and inversely, t_A can be determined from the rotation coefficients by integrating

$$\dot{t}_A = -\rho t_A - \sigma^* t_A^*. \tag{4.5}$$

ρ may also be written as $\rho = -\dot{g}_{AB}g^{AB}/4$. In terms of ρ (ray divergence), σ (ray shear) and τ, differential invariants for the inner geometry may be written down, which could be used as invariant descriptions of the cone geometry (see Dautcourt (1971) for their use as characteristic initial data on two intersecting null hypersurfaces). Notice that the Gaussian curvature,

$$K = -2\tau\tau^* + t^A\tau_{,A} + t^{A^*}\tau^*_{,A}, \tag{4.6}$$

is, in general, no inner differential invariant. Other descriptions of the inner cone geometry are given as projections of the Weyl tensor and Ricci tensor into the cone:

$$\tfrac{1}{2}R_{\mu\nu}p^\mu p^\nu = \omega, \qquad C_{\mu\nu\rho\sigma}p^\mu p^\rho t^{\nu *} t^{\sigma *} = \Psi \tag{4.7}$$

(t^μ is a four-dimensional complex vector orthogonal to $p^\mu = v_{,\nu}g^{\mu\nu}$ which corresponds to the complex direction t^A in the cone). In terms of the rotation coefficient,

$$\omega = \dot\rho - \rho^2 - \sigma\bar\sigma, \qquad \Psi = \dot\sigma - 2\rho\sigma. \tag{4.8}$$

The behaviour of the cone geometry near the vertex $u = 0$ follows from expanding (4.2) in powers of u:

$$g_{AB} = \tfrac{1}{2}u^2 \underset{2}{g_{AB}} + u^4 \underset{4}{g_{AB}} + \ldots \tag{4.9}$$

(no terms $\sim u^3$).

The transversal coordinates w^A will be chosen to make the cone metric near the observer (essentially $\underset{2}{g_{AB}}$) as simple as possible. We want to identify w^A with the reference angles θ and ϕ at the observer's sky. A general null direction $\bar p^\mu$ at P (with components taken in the normal coordinate system) can be parametrised by

$$\bar p^i = \bar p^0 n^i, \qquad n^i = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta). \tag{4.10}$$

Then from (4.1) and (4.2)

$$\tfrac{1}{2}u^2 \underset{2}{g_{AB}}\, \mathrm{d}w^A\, \mathrm{d}w^B = u^2(\bar p^0)^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2). \tag{4.11}$$

The arbitrariness of $\bar p^0$ is connected with the allowed change $u \to u' = \alpha(w^A)u$ of the affine parameter, u; for $\bar p^0 \to \bar p^{0\prime} = p^0/\alpha$ does not change the inner metric. Let a Lorentz transformation of the normal coordinate system change $\bar x^0$ into the observer's proper time. If $\bar V^\mu$ is his velocity, we have $\bar V^0 = 1$. The condition $\bar V^\mu \bar p_\mu = -1/\sqrt{2}$ or equivalently $\bar p^0 = 1/\sqrt{2}$ fixes the gauge of u. An observer determines an affine parameter on his light cone uniquely. The next expansion term u^4 in (4.9) involves the local Riemann tensor at P:

$$\underset{4}{g_{AB}} = \tfrac{1}{3}\bar p^\mu_{,A}\bar p^\nu_{,B}\bar R_{\mu\rho\nu\sigma}\bar p^\rho \bar p^\sigma. \tag{4.12}$$

A short calculation shows that

$$\underset{4}{g_{AB}} = \tfrac{1}{3}\Psi_0 \underset{1}{t_A}\underset{1}{t_B} + \tfrac{1}{3}\Psi_0^* \underset{1}{t_A^*}\underset{1}{t_B^*} + \tfrac{1}{6}\omega_0 \underset{2}{g_{AB}} \tag{4.13}$$

where Ψ_0, ω_0 are the values for $u \to 0$. t_A, ρ and σ behave as

$$t_A = \underset{1}{t_A}u + \ldots, \qquad \rho = -u^{-1} + \omega_0 u/3 + \ldots, \qquad \sigma = \tfrac{1}{3}\Psi_0 u/3 + \ldots \tag{4.14}$$

near the vertex. Subsequently, we use the subscript v in ρ_v, σ_v, to denote the rotation coefficients of the past cone $v = 0$.

5. Cone initial value problem for vacuum fields

Now consider the initial value problem on the past light cone. A straightforward calculation gives the components of the Riemann tensor and Ricci tensor in null coordinates (partly printed in the appendix).

We first consider vacuum fields, $R_{\mu\nu}=0$. For the calculation of the vacuum fields inside the past light cone $v=0$ with data specified on the past light cone, not all ten field equations need to be solved inside the past cone. As in the case of the characteristic initial-value problem based on two intersecting null hypersurfaces (Sachs 1962, Dautcourt 1963, 1967), the four Bianchi identities

$$G^{\mu\nu}_{;\nu}=0 \tag{5.1}$$

allow four field equations to be eliminated. The relevant lemma can be formulated as follows:

Lemma 1. If the four vacuum field equations

$$g^{AB}R_{AB}=0, \qquad R_{0A}=0, \qquad R_{00}=0 \tag{5.2}$$

hold on $C^-(P)$ and if the remaining six equations

$$R_{01}=0, \qquad R_{AB}-\tfrac{1}{2}g_{AB}R_{CD}g^{CD}=0, \qquad R_{11}=0, \qquad R_{1A}=0 \tag{5.3}$$

are satisfied in $D^-(P)$, then equations (5.2) are also satisfied in $D^-(P)$.

A simple proof runs as follows. As a consequence of (A16)–(A18) the Bianchi identities (5.1) may be written, if the equations (5.3) are taken into account, as

$$R'_{00}=f_0(R_{00},R_{0B},R_{AB}g^{AB}), \qquad (g^{AB}R_{AB})'=f_1(R_{00},R_{0B},R_{AB}g^{AB}),$$
$$R'_{0A}=f_A(R_{00},R_{0B},R_{AB}g^{AB}) \tag{5.4}$$

(the prime denotes $\partial/\partial v$). The functions on the right-hand side depend linearly on the components R_{00}, R_{0B}, $R_{AB}g^{AB}$; they also involve the intrinsic (as to the cone $v=0$) derivatives $\partial/\partial u$ and $\partial/\partial w^A$. It is seen from this structure of (5.4) that the exterior derivatives $\partial/\partial v$ of all orders applied to R_{00}, $R_{AB}g^{AB}$, R_{0B} vanish at $C^-(P)$, if these quantities vanish by themselves. Thus, in the analytic case, the relations (5.2) hold in a finite neighbourhood $v\geqslant 0$ of $C^-(P)$ (care is necessary, since the coordinate system breaks down in a finite affine distance v^* from $C^-(P)$). The derivatives $\partial/\partial v$ on the left-hand side of (5.4) are to be interpreted as limits $\lim_{v\to+0}(\partial f/\partial v)=f^{(+)'}$, involving only positive values of v (the interior of the past cone corresponds to $v>0$ in our convention). Thus the validity of (5.2) is confined formally to the interior of the past cone. One might equally well interpret the v derivatives at $C^-(P)$ in (5.2) as limits from negative v: $\lim_{v\to-0}(\partial f/\partial v)=f^{(-)'}$, and again conclude—in the case of vacuum—that the 'v-minus derivatives' of R_{00}, $R_{AB}g_{AB}$, R_{0B} vanish to all orders also for $v<0$, $u>0$, if they vanish at $v=0$, provided the remaining equations (5.3) are satisfied on $C^-(P)$.

As shown in § 6, there is a stringent difference between the cases, because in the latter case the initial data on the future cone $C^+(P)$ must also be given for a complete solution of the initial value problem (indeed, suitable initial data given on the full cone determine the gravitational field also in the relativistic presence of P, cf Penrose 1963). The interior $D^+(P)$ of the future light cone cannot, however, be reached with the development (5.4). Here a complementary lemma holds, which is noted for completeness.

Lemma 2. If the four vacuum field equations

$$R_{11}=0, \qquad g^{AB}R_{AB}=0, \qquad R_{1A}=0, \tag{5.5}$$

hold on the future cone $C^+(P)$, and if the remaining equations

$$R_{01}=0, \qquad R_{AB}-\tfrac{1}{2}g_{AB}R_{CD}g^{CD}=0, \qquad R_{00}=0, \qquad R_{0A}=0, \tag{5.6}$$

hold in $D^+(P)$, then (5.5) hold also in $D^+(P)$.

Lemma 1 stated above separates the equations in the six propagation equations (5.3), which must be satisfied everywhere on the manifold, and in the four cone equations ('hypersurface equations') (5.2), which need to be satisfied on the cone only. (5.3) has just the right number of equations to propagate the six unknown field functions h, k_A, g_{AB} into the past cone. The four cone equations (5.2) restrict six starting values of h, k_A, g_{AB} (or their derivatives), so essentially only two generic functions need to be given on the past cone in the vacuum case. The cone equations may be written in a simplified form, with the values for the metric tensor taken on the past cone:

$$R_{00}\equiv -\tfrac{1}{2}g^{AB}\ddot{g}_{AB}-\tfrac{1}{4}\dot{g}^{AB}\dot{g}_{AB}=0, \tag{5.7}$$

$$R_{0A}\equiv\tfrac{1}{2}\dot{k}'_A-\tfrac{1}{2}\dot{g}_{BC|A}g^{BC}+\tfrac{1}{2}\dot{g}_{AB|C}g^{BC}+\tfrac{1}{4}k'_A\dot{g}_{BC}g^{BC}=0, \tag{5.8}$$

$$g^{AB}R_{AB}\equiv\bar{R}_{CADB}g^{CD}g^{AB}-k'_{A|B}g^{AB}-\tfrac{1}{2}k'_Ak'_Bg^{AB}+4\dot{\rho}_u+2\dot{g}_{AB}g^{AB}\rho_u=0. \tag{5.9}$$

In (5.9) we have introduced the quantity

$$\rho_u=\tfrac{1}{4}g^{AB}g'_{AB} \tag{5.10}$$

which represents the divergence of the cone-like null hypersurfaces $u=\text{constant}$ at their intersection with the past cone $v=0$. (The different sign compared with ρ_v in § 4 arises because v increases inside the past light cone, that is, in the direction of the apparent vertex of the cone-like hypersurfaces $u=\text{constant}$). The remaining equations (5.3) must be written out in full if the propagation problem is considered. Here we are interested in the gravitational field on the past cone. This allows us to introduce the simplifications (2.9) for all components of the Ricci tensor. The equations (5.3)—arranged in a certain order and partly combined with (5.2) to simplify the integration—then take the form (the Gaussian curvature of the two-dimensional surfaces $u=\text{constant}$ on C^- is denoted by K)

$$R_{AB}=\dot{g}'_{AB}-\rho_v g'_{AB}+\rho_u\dot{g}_{AB}-\tfrac{1}{2}(g'_{AC}\dot{g}_{BD}g^{CD}+g'_{BC}\dot{g}_{AD}g^{CD})$$
$$-\tfrac{1}{2}(k'_{B|A}+k'_{A|B})-\tfrac{1}{2}k'_Ak'_B+Kg_{AB}, \tag{5.11}$$

$$R_{01}+\tfrac{1}{2}R_{AB}g^{AB}=\dot{h}'-4\rho_u\rho_v+K-\tfrac{3}{4}k'_Ak'_Bg^{AB}+\tfrac{1}{4}g^{AB'}\dot{g}_{AB}, \tag{5.12}$$

$$R_{11}=-2\rho'_u-2\rho_uh'+\tfrac{1}{4}g^{AB'}g'_{AB}, \tag{5.13}$$

$$R_{A1}=-\tfrac{1}{2}k''_A-\tfrac{1}{2}k'_AH'+\tfrac{1}{2}h'_{,A}+\tfrac{1}{2}g'_{AB|C}g^{BC}+\tfrac{1}{2}g'_{AB}g^{BC}k'_C-\rho_uk'_A-2\rho_{u,A}. \tag{5.14}$$

(Note (5.9) is contained in (5.11). For the propagation problem, $R_{AB}-\tfrac{1}{2}g_{AB}R_{CD}g^{CD}=0$ instead of $R_{AB}=0$ should be taken as propagation equation.)

6. Integration of the field equations down the cone

The form of equations (5.7)–(5.14) suggests a definite integration scheme, which starts with (5.7). This equation (equivalent to the first equation in (4.7)) is the only restriction for the cone geometry imposed by the field equations. (5.7) may be used to determine g_{AB} completely, if two components of g_{AB} (say the 'conformal' metric g_{AB}, that is g_{AB} up to a factor) are given. For instance, if

$$g_{AB}=\tfrac{1}{2}fu^2\gamma_{AB} \tag{6.1}$$

is assumed with $|\gamma_{AB}|=\sin^2\theta$, (5.7) gives an ordinary differential equation for the conformal factor $f=f(u, w^A)$:

$$\ddot{f}/f+2\dot{f}/uf-\dot{f}^2/2f^2=-R_{00}+\tfrac{1}{4}\dot{\gamma}_{AB}\dot{\gamma}^{AB} \tag{6.2}$$

which can be solved, provided γ_{AB} is given. f remains finite $(=1)$ at the vertex $u=0$. Other possibilities are to give the shear σ_v or to specify the Penrose function Ψ_v as function of u, θ and ϕ. The disadvantage of all these data is their gauge dependence. The use of gauge-independent differential invariants of the null geometry as initial data was discussed by Dautcourt (1971). The data proposed there show singularities on otherwise completely regular submanifolds of the cone. Thus the problem of giving two suitable functions of the intrinsic cone geometry remains open. With known g_{AB}, equation (5.8) can be solved for the k'_A:

$$k'_A=\frac{1}{\sqrt{g}}\int_0^u \sqrt{g}\,\mathrm{d}u\,(-2R_{0A}+\dot{g}_{BC|A}g^{BC}-\dot{g}_{AB|C}g^{BC}). \tag{6.3}$$

At the lower boundary $u=0$ the integral (6.3) becomes formally singular, since $\sqrt{g}\sim u^2\sin\theta/2$ for small u. To guarantee that $k'_A\sim u^2$ for small u, as follows from (3.5), the bracket in (6.3) should vanish like u for $u\to 0$. This is indeed true: a non-singular tensor like the matter tensor replacing $R_{\mu\nu}$ has components R_{0A} which vanish at least like u in the null coordinate system; also the rest of the integrand vanishes like u as can be checked by direct calculation.

Singular integrals of the type (6.3) often occur in our treatment due to the vertex singularity, but turn out to be finite (one could use 'renormalised' quantities like g_{AB}/u^2 or more generally, $g_{AB}/(u-v)^2$, to make the regular behaviour more transparent). From (5.9) we calculate the divergence of the null hypersurfaces $u=$ constant at their intersection with the past cone $v=0$:

$$\rho_u=\frac{1}{4\sqrt{g}}\int_0^u \sqrt{g}\,\mathrm{d}u\,(-2K+k'_{A|B}g^{AB}+\tfrac{1}{2}k'_Ak'_Bg^{AB}+R_{AB}g^{AB}). \tag{6.4}$$

Since R_{AB} as well as k'_A vanish like u^2 for small u, the last three terms of the bracket in (6.4) remain constant for $u\to 0$. They contribute to ρ_u a term which vanishes for $u\to 0$. The first term involving the Gaussian curvature $K\approx 2/u^2$ produces the expected behaviour $\rho_u\approx -1/u$ near the vertex.

The remaining content of (5.11) may be represented as a simple linear differential equation for the shear σ_u of the null hypersurfaces $u=$ constant:

$$\dot{\sigma}_u-\rho_v\sigma_u=-\rho_u\sigma_v-\tfrac{1}{2}t^{A*}t^{B*}(R_{AB}+k'_{A|B}+\tfrac{1}{2}k'_Ak'_B). \tag{6.5}$$

As usual, the right-hand side of (6.5) is known from previous integrations. σ_u follows as

$$\sigma_u = -g^{-1/4}\int_0^u g^{1/4}\,\mathrm{d}u\;(\sigma_v\rho_u + \tfrac{1}{2}t^{A^*}t^{B^*}[R_{AB} + k'_{A|B} + \tfrac{1}{2}k'_A k'_B]). \tag{6.6}$$

The integrand in (6.6) is proportional to u near the vertex, thus $\sigma_u \sim u$ for small u. Equation (5.12) determines h' from its value $\lim h' \to 0$ for $u \to 0$ ($h' \sim u$ near the vertex)

$$h' = \int_0^u \mathrm{d}u\;(R_{01} + \tfrac{1}{2}R_{AB}g^{AB} + \tfrac{3}{4}k'_A k'_B g^{AB} - K + 4\rho_u\rho_v - \tfrac{1}{4}g^{AB'}\dot{g}_{AB}). \tag{6.7}$$

In order to guarantee that h' tends to zero for $u \to 0$, the integrand must reach a finite value for $u \to 0$.

Indeed, the divergences $1/u^2$ which are present in the last three terms of (6.7) cancel. The last equations (5.13) and (5.14) determine the second derivatives ρ'_u and k''_A algebraically in terms of known quantities:

$$\rho'_u = -\tfrac{1}{2}R_{11} - \rho_u h' + \tfrac{1}{8}g^{AB'}g'_{AB}, \tag{6.8}$$

$$k''_A = -2R_{1A} - k'_A(2\rho_u + h') + k'_C g'_{AB}g^{BC} - 4\rho_{u,A} + h'_{,A} + g'_{AB|C}g^{BC}. \tag{6.9}$$

The still missing second-order derivatives of the metric tensor are h'' and $\sigma'_u \cdot h''$ is obtained from $(R_{01} + \frac{1}{2}g^{AB}R_{AB})'$ and σ'_u from $(R_{AB} - \frac{1}{2}g_{AB}R_{CD}g^{CD})'$. The calculation proceeds as in the previous cases. By means of these formulae, all second derivatives of the metric tensor are known on the past light cone. The Riemann tensor (and its invariants) can, therefore, be constructed on $C^-(P)$ (cf equations (A3)–(A12)) and the local gravitational field on C^- is known.

This integration scheme extends to all derivatives of the Riemann tensor. Taking the v derivatives of $R_{AB} - \frac{1}{2}g_{AB}R_{CD}g^{CD}$, R_{01}, R_{11} and R_{1A}, one is able to determine higher-order derivatives of h, k_A and g_{AB} at $v = 0$, and so to construct an analytic solution $g_{\mu\nu}$ as a power series in v. The complexity of the resulting equations rapidly increases, thus one should either consult a formula manipulation system or turn to a null tetrad formalism, which allows a more transparent derivation on the cost of increasing the number of variables. In any case, the whole procedure has to be turned into a computer program to solve the past light cone initial-value problem numerically. It was stated in § 5 that the gravitational field in the 'relativistic presence' $u > 0$, $v < 0$ cannot be calculated from initial data on the past cone $C'(P)$. We may explain this qualitatively from the structure of the v-derived propagation equations (5.3), which represent ordinary differential equations in the independent variable u for the higher-order v derivatives of h, k_A and g_{AB} at $v = 0$ as dependent variables. Remember that v derivatives of any function f on the past cone $v = 0$ are usually to be taken as $\lim_{v\to+0}\partial f/\partial v = f^{(+)'}$. Solving these equations by integrating down the light cone requires specification of initial data for $u \to 0$. Now for $u \to 0$, v derivatives $f^{(+)'}$ turn into u derivatives $\dot{f}^{(+)}$ along the opposite direction of the observer's sky ($\theta \to \pi - \theta$, $\phi \to \phi + \pi$). Thus the initial values are known from previously calculated quantities on the past cone. If the gravitational field for $u > 0$, $v < 0$ is considered, any higher-order v derivative must be interpreted as $f^{(-)'} = \lim_{v\to-0}\partial f/\partial v$. Their initial values for $u \to 0$ are v derivatives taken on the future light cone $u = 0$. Thus data on the future light cone need to be known. The discussion shows that the same type of initial data must be given as on the past cone.

7. Extension to incoherent matter

The integration procedure described in § 6 is easily generalised to the case where matter is present. For simplicity, incoherent matter

$$T_{\rho\sigma} = \mu V_\rho V_\sigma \tag{7.1}$$

is assumed. Lemma 1 (allowing a reduction of the number of field equations which must be solved) now reads as follows.

Provided the four equations

$$R_{AB} g^{AB} - 2\varkappa\mu V_0 V_1 = 0, \qquad R_{0A} - \varkappa\mu V_0 V_A = 0, \qquad R_{00} - \varkappa\mu V_0^2 = 0 \tag{7.2}$$

are satisfied on $C^-(P)$, and the six equations

$$\begin{aligned} &R_{01} - \varkappa\mu V_0 V_1 + \tfrac{1}{2}\varkappa\mu = 0, \\ &R_{AB} - \tfrac{1}{2} g_{AB} R_{CD} g^{CD} - \varkappa\mu[V_A V_B + g_{AB}(\tfrac{1}{2} - V_0 V_1)] = 0, \\ &R_{11} - \varkappa\mu V_1^2 = 0, \qquad R_{1A} - \varkappa\mu V_1 V_A = 0, \end{aligned} \tag{7.3}$$

as well as the four divergence relations $T^{\rho\sigma}_{;\sigma} = 0$ or

$$V^\rho_{;\sigma} V^\sigma = 0, \tag{7.4}$$

$$\mu_{,\rho} V^\rho / \mu + V^\rho_{;\rho} = 0, \tag{7.5}$$

are satisfied on $D^-(P)$, then (7.2) hold also on $D^-(P)$. To (7.4) and (7.5), the relation $V^\mu V_\mu = -1$, or

$$-2 V_0 V_1 + V_A V_B g^{AB} = -1 \tag{7.6}$$

on $C^-(P)$, must be added. The proof in § 5 extends to the present case. The construction of solutions starting from the past light cone initial data proceeds as in § 6 (the components $R_{\mu\nu}$ must be replaced by the corresponding matter quantities instead of putting $R_{\mu\nu} = 0$).

The relations $T^{\rho\sigma}_{;\sigma} = 0$ do not restrict the initial values of the matter density μ and velocity components V^ρ on $C^-(P)$. Instead, (7.3) and (7.4) allow us to express the exterior derivatives μ', V_1' or V_0', V_A' in terms of the metric components on $v =$ constant.

Calculated on $C^-(P)$, we have:

$$V_0' = V_{0,A} V^A / V_0 - \dot{V}_0 V_1 / V_0 - \dot{g}_{AB} V^A V^B / (2 V_0), \tag{7.7}$$

$$V_1' = V_{1,A} V^A / V_0 - \dot{V}_1 V_1 / V_0 - V_1 h' + V_1 V^A k_A' / V_0 - g_{AB}' V^A V^B / (2 V_0), \tag{7.8}$$

$$V_A' = (V_{A,B} V^B - \dot{V}_A V_1 - g_{BC,A} V^B V^C / 2) / V_0, \tag{7.9}$$

$$\mu' = (\mu\theta + \mu_{,A} V^A) / V_0 + \dot{\mu} V_1, \tag{7.10}$$

with the expansion velocity

$$\theta = -V_0' - \dot{V}_1 + V_{A|B} g^{AB} + k_A' V^A - 2\rho_u V_0 + 2\rho_v V_1. \tag{7.11}$$

(Because $V^\mu V_\mu = -1$, (7.7)–(7.9) are not independent.) The integration scheme discussed in the previous sections shows that—apart from quantities describing the matter tensor—only two geometrical quantities on the past light cone $C^-(P)$ must be given to determine a local world model. A possible candidate is the complex Penrose function Ψ. Additionally we must specify (in the case of incoherent matter) the matter density $\mu(u, \theta, \phi)$ and three velocity components—say $V_0(u, \theta, \phi)$ and $V_A(u, \theta, \phi)$—on $C^-(P)$. Up to a factor $1/\sqrt{2}$, V_0 is equal to the redshift $1+z$ of light emitted at the world point $(u, v=0, \theta, \phi)$ from matter moving there with the velocity V^μ. V_A can be considered as the proper motion components of this matter. The six real functions in Ψ, μ, V_0 and V_A cannot be specified as completely arbitrary functions of u, θ, ϕ on $C^-(P)$. The vertex singularity (as well as the various caustics) impose additional conditions. Near the vertex $u=0$, μ tends to an arbitrary finite value, V_0 to $1/\sqrt{2}$, and V_A vanishes at least as u for $u \to 0$. The velocity components are given here in the observer's null coordinate system. A characterisation of the fluid motion independently of the observer is provided by the kinematical invariants expansion velocity θ (given in (7.11)), shearing velocity $(\Sigma_{\mu\nu} = \Sigma_{\nu\mu})$ and rotation velocity $(\Omega_{\mu\nu} = -\Omega_{\nu\mu})$, obtained from the algebraic decomposition of $V_{\mu;\nu}$:

$$V_{\mu;\nu} = \theta(g_{\mu\nu} + V_\mu V_\nu)/3 + \Sigma_{\mu\nu} + \Omega_{\mu\nu} - V_{\mu;\rho} V_\nu V^\rho. \tag{7.12}$$

In our case of vanishing pressure the last term vanishes. To characterise the type of motion on the past light cone, the scalars $\Sigma = (\Sigma_{\mu\nu}\Sigma^{\mu\nu}/2)^{1/2}$ and $\Omega = (\Omega_{\mu\nu}\Omega^{\mu\nu}/2)^{1/2}$ besides θ should be used, calculated in the null coordinate system. They contain first exterior derivatives of the metric tensor components which are already calculated in (6.3)–(6.7). Thus the kinematical invariants on the past light cone can be expressed in terms of characteristic initial data.

8. Simple examples

We consider first the static Einstein universe with line element

$$ds = -d\tau^2 + d\chi^2 + \sin^2\chi(\sin^2\theta\, d\phi^2 + d\theta^2). \tag{8.1}$$

The transformation $\tau = -(u+v)/\sqrt{2}$, $\chi = (v-u)/\sqrt{2}$ introduces null coordinates:

$$ds^2 = -2du\, dv + \sin^2([v-u]/\sqrt{2})(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{8.2}$$

The geometry of this space–time is extremely simple. The past light cone $v=0$ has the divergence $\rho_v = -\cot(u/\sqrt{2})/\sqrt{2}$, shear $\sigma_v = 0$ and Gaussian curvature $K = 1/\sin^2(u/\sqrt{2})$ of the wave surfaces $u=$ constant. For $v = \sqrt{2}\pi$ the rays meet again in a second non-singular vertex. $C^-(P)$ covers the whole past light cone. We may consider an initial-value problem on $C^-(P)$ starting with the data

$$\Psi = 0, \qquad V_0 = 1/\sqrt{2}, \qquad V_A = 0, \qquad \mu = \text{constant}, \tag{8.3}$$

on $v=0$. We would have to extend the treatment given so far, since a pressure $p = 2/\varkappa - \mu =$ constant and a constant lambda $\Lambda = 1 + \varkappa p$ must be assumed for this model. It suffices to replace $R_{\mu\nu}$ in (6.2)–(6.9) and other relations by

$$R_{\mu\nu} = \varkappa V_\mu V_\nu(\mu + p) + g_{\mu\nu}(\Lambda + \tfrac{1}{2}\varkappa[\mu - p]). \tag{8.4}$$

$\Psi = 0$ on $v=0$ requires $\sigma_v = 0$: From (4.7) a solution $\sigma_v = a/\sqrt{g}$, $\dot{a} = 0$, would be admitted, but σ_v diverges for $u \to 0$ contrary to (4.13). So $a=0$, and the first equation (4.7) can be integrated with $\omega = \frac{1}{2}$ and $\sigma_v = 0$. The result is $\rho_v = -(1/\sqrt{2})\cot(u/\sqrt{2})$. The metric on the cone follows as $ds^2 = \sin^2(u/\sqrt{2})(d\theta^2 + \sin^2\theta d\phi^2)$ by integrating (4.5) and using (4.3). The further integrations can be carried out step by step, yielding $\rho_u = -\cot(u/\sqrt{2})/\sqrt{2}$ from (6.4), $\sigma_u = 0$ from (6.6), $h' = 0$ from (6.7), $\rho'_U - 1/(2\sin^2(u/\sqrt{2}))$ from (6.8) and $k''_A = 0$ from (6.9). We stop here since the propagation problem is not treated in this article. The method may not be very useful to obtain analytic solutions of the field equations. Its value is the unique characterisation of a cosmological model by few data such as (8.3), data which are closely related to observations (for instance, no redshift will be seen from galaxies at rest in the cosmic matter).

As a second example we use the Friedman models. A transformation from the coordinates t, χ, θ, ϕ employed in the Friedman metric

$$ds^2 = -dt^2 + R^2(t)(d\chi^2 + S^2(\chi)\,d\Omega^2), \qquad d\Omega^2 = \sin^2\theta\, d\phi^2 + d\theta^2, \tag{8.5}$$

with

$$S(\chi) = \begin{cases} r_0 \sin(\chi/r_0), \\ \chi, \\ r_0 \sinh(\chi/r_0), \end{cases} \qquad \text{for } k = 1, 0, -1 \text{ respectively}$$

and $r_0 = [(2q_0 - 1)\chi\mu_0/(6q_0 k)]^{-1/2}$ (μ_0 and q_0 are the present matter density and present deceleration parameter) to null coordinates u, v, θ, ϕ can be carried out using the following transformation. First change t to τ by

$$\tau = \int_{t_0}^{t} \frac{dt}{R(t)}$$

($\tau < 0$ in the past) and denote $R(t(\tau)) = a(\tau)$. For the present time $\tau = 0$ we may assume $a(0) = 1$. Notice that the coordinates χ, τ carry a length dimension whereas R or a does not. Null coordinates $\tilde{u}$, $\tilde{v}$ may again be introduced by

$$\tilde{v} = -(\chi + \tau)/\sqrt{2}, \qquad \tilde{u} = (\chi - \tau)/\sqrt{2},$$

resulting in

$$ds^2 = -2\tilde{h}\, d\tilde{u}\, d\tilde{v} + \tilde{h}S^2\, d\Omega^2 \tag{8.6}$$

with

$$\tilde{h} = a^2(-[\tilde{u} + \tilde{v}]/\sqrt{2}), \qquad S = S([\tilde{u} - \tilde{v}]/\sqrt{2}).$$

However, $\tilde{u}$ and $\tilde{v}$ are not yet affine parameters on the light cones $\tilde{v} = 0$ and $\tilde{u} = 0$. To obtain the (uniquely determined) particular type of null coordinates employed in this article, define a function

$$p(\xi) = \int_0^{\xi} a^2(-\xi)\, d\xi \tag{8.7}$$

and its inverse $\xi = \xi(p)$. Then the coordinate transformation

$$u = \sqrt{2}p(\tilde{u}/\sqrt{2}), \qquad v = \sqrt{2}p(\tilde{v}/\sqrt{2})$$

transforms (8.6) into

$$ds^2 = -2h\, du\, dv + l\, d\Omega^2 \tag{8.8}$$

with

$$h(u, v) = \frac{a^2(-\xi[u/\sqrt{2}] - \xi[v/\sqrt{2}])}{a^2(-\xi[u/\sqrt{2}])a^2(-\xi[v/\sqrt{2}])}, \tag{8.9}$$

$$l(u, v) = a^2(-\xi[u/\sqrt{2}] - \xi[v/\sqrt{2}])S^2(\xi[u/\sqrt{2}] - \xi[v/\sqrt{2}]). \tag{8.10}$$

Since $\xi(0) = 0$, $h \to -1$ on the future cone $u = 0$ as well as on the past cone $v = 0$. We are interested in obtaining the characteristic initial data on the past light cone $v = 0$. Since g_{AB} is conformal to the sphere metric with conformal factor independent of the transversal coordinates θ and ϕ, we have vanishing σ_v as well as $\Psi_v = 0$ on $C^-(P)$. The velocity components V^μ in null c oordinates are obtained transforming $V^\mu = \delta^\mu_0$ from the original coordinate system (8.5). For the matter density, $\mu R^3 = \mu_0 R_0^3$ is used. A straightforward calculation shows that a Friedman universe is uniquely characterised by the past light cone initial data

$$\Psi_v = 0, \qquad V_0 = Z(u)/\sqrt{2}, \qquad V_A = 0, \qquad \mu = \mu_0 Z^3(u). \tag{8.11}$$

$Z(u)$ is the redshift function $1 + z(u)$ depending on the affine parameter u. Explicitly we may express $Z(u)$ in terms of a function $f(x)$ (for $k \neq 0$):

$$Z(u) = \begin{cases} (2q_0 - 1)/(2q_0 k) f[x_0 - u/r_0] & k \neq 0, \\ (1 - u/u_\infty)^{-2/5}, & k = 0. \end{cases} \tag{8.12}$$

$f(x)$ is defined by its inverse function $x(y)$:

$$x(y) \equiv \int_y^\infty \frac{\mathrm{d}y}{y^3 (y - k)^{1/2}}$$

$$= \begin{cases} \frac{3}{8}\pi - \dfrac{(3y+2)}{4y^2}(y-1)^{1/2} - \frac{3}{4}\tan^{-1}(y-1)^{1/2}, & k = 1, \\ \dfrac{(2-3y)}{4y^2}(\sqrt{y}+1)^{1/2} - \frac{3}{8}\ln\left(\dfrac{(y+1)^{1/2}-1}{(y+1)^{1/2}+1}\right), & k = -1. \end{cases} \tag{8.13}$$

Since $f(x(y)) \equiv y$ by definition, it is seen that $Z(0) = 1$, if we put $x_0 = x[2q_0 k/(2q_0 - 1)]$. A particular Friedman model is characterised by its familiar parameters μ_0 (present matter density), q_0 and $H_0 = [\varkappa\mu_0/(6q_0)]^{1/2}$. Note that the redshift z increases monotonically with u until it reaches infinity for a value u_∞ of the affine parameter, given by

$$\begin{aligned} u_\infty &= x_0 (6/\varkappa\mu_0)^{1/2} [2q_0 k/(2q_0 - 1)]^{5/2} && \text{for } k \neq 0, \\ u_\infty &= (24/25\varkappa\mu_0)^{1/2} && \text{for } k = 0. \end{aligned} \tag{8.14}$$

It is in general not easy to determine the characteristic initial data for a given more complicated cosmological model (equivalent to, and to some degree identical to the determinination of its observational content), because in general one needs to integrate the null geodesics.

On the other hand, one is free to generalise the initial (8.11) for the Friedman universe by, say, introducing proper motion or a shear $\sigma_v \neq 0$, without getting consistency problems. However, properties of homogeneity and universal isotropy may be destroyed by this procedure. It is trivial to formulate the initial data as being isotropic for the Earth observer, but any other sufficiently distant observer comoving with dust matter ('privileged' observer) will in general find no isotropy (this can be checked for observers on or inside the Earth observer's past light cone. But one expects this violation of a Copernican principle also for any other privileged observer, no mattter how the missing data on the future cone are chosen). To express homogeneity (existence of three-parameter groups of motions acting transitively on space-like hypersurfaces at least in the space–time region $D^-(P)$) in terms of the cone initial data is a problem still to be solved like the corresponding problem for universal isotropy. In both cases one expects that the initial data are fairly strongly restricted by Copernican principles. On the other hand, it is easy to formulate conditions for 'local homogeneity' (homogeneity near the past light cone) as restrictions for the initial data. One could check with the help of the propagation equations if homogeneity is also present inside the past light cone. Eventually, this is all that one can do if one wants to stick exclusively to observations.

9. Trapped surfaces

Because of its importance for the singularity theorems (Hawking and Ellis 1973) we conclude with some remarks on trapped surfaces on the past light cone (cf Hawking and Ellis 1968). Near the vertex P, the divergence ρ_v of the past cone diverges as $-1/u$ for $u \to 0$. If, for example, sufficient matter is present down the rays of the past cone, ρ_v increases and may become positive after going through a 'turning point' $\rho_v = 0$ at $u = u_2$, say. Its geometrical meaning is seen by considering the surface area $A = \int \sqrt{g}\, d\theta\, d\phi$ of the two-dimensional surfaces $u = \text{constant}$, $v = 0$. We have

$$\dot{A} = -2 \int \rho_v \sqrt{g}\, d\theta\, d\phi\,; \tag{9.1}$$

thus the area A stops increasing with u and shrinks after passing a turning point $u = u^*$. Since u_2 contrary to u^* depends in general on the angles θ and ϕ, u^* need not coincide with u_2. (For a Minkowski light cone, $u_2 = u^* \to \infty$.) Apparent images of galaxies and quasars on the sky, after shrinking with u for small u, expand for $u > u_2$ (but still with decreasing surface luminosity). Before the point u_2 with $\rho_v = 0$ is reached, one meets another special point at $u_1 \leq u_2$, where $\rho_v^2 - \sigma_v \sigma_v^*$ becomes zero. Elliptical points on the cone (present for $u < u_1$) change into hyperbolic points (for $u > u_1$).†

Now consider the two surfaces $u = \text{constant}$, $v = \text{constant}$ on the cone-like null hypersurfaces $u = \text{constant}$ intersecting with the past cone. The divergence ρ_u is a measure of how rapidly the hypersurfaces $u = \text{constant}$ converge into the past at its intersection with the past cone.

Near the vertex $\rho_u \approx -1/u$, and $\rho_u \rho_v \approx 1/u^2 > 0$. Provided ρ_u does not change its sign down the cone, $\rho_u \rho_v$ becomes negative for $u > u_2(\theta, \phi)$. If this holds on the whole closed two-surface Σ_2 (with equation $u = u_2(\theta, \phi)$), Σ_2 is called a trapped surface. Both past directed sets of light rays orthogonal to Σ_2 converge for $u > u_2$ respectively $v > 0$ (remember that u and v both increase towards the past). The picture changes, if ρ_u also changes its sign at $u = u_3(\theta, \phi)$. If $u_3 < u_2$, the surface $u = u_3$ on C^- is a trapped surface with regard to future directed rays. For $u_3 > u_2$ the surfaces $u = \text{constant}$ could be trapped with regard to both future and past directed rays. From (6.4) it is not clear if ρ_u increases in all cases for increasing u. More stringent conclusions can be derived with the concept of an average trapped surface (Hartle and Wilkins 1973). This is a compact space-like two-surface with orthogonal light rays which generate wavefronts (space-like two-surfaces) of decreasing area for both outward and inward directed rays. Consider the change $\partial A/\partial v \equiv A'$ of the surface area A of Σ on the past cone as a function of u. $A' < 0$ holds near the vertex as well as $\dot{A} > 0$, while $\dot{A}A' > 0$ is required for an averaged trapped surface. For the change of A' with u one derives

† The two types of points show a different behaviour of the change of distances to the next light rays, cf Dautcourt (1967).

from (6.4)

$$\dot{A}' = -\int K\sqrt{g}\,\mathrm{d}\theta\,\mathrm{d}\phi + \tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g}(k'_A g^{AB})_{|B}$$

$$+\tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g}(R_{AB}g^{AB} + \tfrac{1}{2}k'_A k'_B g^{AB}). \tag{9.2}$$

The surface integral over the Gaussian curvature gives 4π according to the Gauss–Bonnet theorem. The second integrand in (9.2) vanishes, because the integrand represents a divergence and k'_A is regular on the surface. Hence

$$\dot{A}' = -4\pi + \tfrac{1}{2}\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g}(R_{AB}g^{AB} + \tfrac{1}{2}k'_A k'_B g^{AB})$$

or

$$A' = -4\pi u + \tfrac{1}{2}\int_0^u \mathrm{d}u \int \mathrm{d}\theta\,\mathrm{d}\phi\,\sqrt{g}(R_{AB}g^{AB} + \tfrac{1}{2}k'_A k'_B g^{AB}) \tag{9.3}$$

after integrating down the cone. Since the integrand is always positive, the integral term in (9.3) could eventually compensate the negative first term for a sufficiently large u. It depends on the specific characteristic initial data, however, whether A' or $\dot{A}$ changes sign, if at all.

We briefly consider two simple cosmological models with regard to trapped surfaces. For the static Einstein universe one derives from (8.2) along the past light cone $v = 0$

$$\dot{A} = -A' = (4\pi/\sqrt{2})\sin(\sqrt{2}u); \tag{9.4}$$

thus $\dot{A}A' < 0$ everywhere on the past cone, apart from the two-surface $u = u^* = \pi/\sqrt{2}$, where both $\dot{A}$ and A' (and already ρ_v and ρ_u) change sign and become zero instantaneously. After this turning point the inside of the past light cone from P becomes the outside of the future cone from the antipodal vertex $\tilde{P}$ ($\tilde{P}$ is reached when all past directed null geodesics through P meet again). Therefore, the two-surfaces $u = \text{constant} > u^*$ may be considered as trapped with regard to both future and past directed rays. The situation is different for the Friedman models. Because of the isotropy of the models, here also the average trapped surfaces coincide with a closed trapped surface $u = \text{constant}$ on the past cone determined by a changing sign of $\rho_u\rho_v$ along a ray. For simplicity, the Einstein–deSitter model is taken. With the function $f(u) \equiv (1 - u/u_\infty)^{-1/5}$, ρ_u and ρ_v can be represented as

$$\rho_u = \frac{-(2-f)}{5u_\infty(1-f)f}, \qquad \rho_v = \frac{-(3f-2)}{5u_\infty f^5(1-f)}. \tag{9.5}$$

f ranges from 1 (at the observer) to 0 (at the past singularity). It is seen that ρ_u cannot become positive within this range of f. However ρ_v does so for $f \leq \frac{2}{3}$ or $u_t \geq 0.868\,u_\infty$. u_t corresponds to the redshift $z = f^{-2} - 1 = 1.25$, where the apparent area of distant sources stops decreasing if one goes back into the past. For $u \geq u_t$, all two-surfaces $u = \text{constant}$ are trapped surfaces. These are trivial examples, but they show different behaviour. It would be interesting to know and to understand the behaviour in more general situations, in particular the influence of rotation and shear motions. Remarkably, the distribution and motion of matter on the past light cone alone suffices to solve this essentially nonlinear problem.

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Appendix

The Riemann tensor in null coordinates

We collect in the appendix a number of formulae connected with the double null coordinates system $x^\mu = (u, v, w^A)$ introduced in § 2. They have been checked using REDUCE (Hearn 1972).

$$\begin{aligned}
&\Gamma_{000} = m\dot{m}, \qquad \Gamma_{001} = \dot{h} - mm', \qquad \Gamma_{00A} = \dot{k}_A - mm_{,A}, \qquad \Gamma_{0A0} = mm_{,A},\\
&\Gamma_{0A1} = \tfrac{1}{2}h_{,A} - \tfrac{1}{2}k'_A, \qquad \Gamma_{0AB} = \tfrac{1}{2}\dot{g}_{AB} + \tfrac{1}{2}(k_{B,A} - k_{A,B}),\\
&\Gamma_{AB0} = \tfrac{1}{2}(k_{A,B} + k_{B,A}) - \tfrac{1}{2}\dot{g}_{AB}, \qquad \Gamma_{AB1} = -\tfrac{1}{2}g'_{AB},\\
&\Gamma_{ABC} = \{ABC\} \equiv \tfrac{1}{2}(g_{AC,B} + g_{BC,A} - g_{AB,C}), \qquad \Gamma_{010} = mm',\\
&\Gamma_{011} = \Gamma_{111} = \Gamma_{11A} = \Gamma_{1A1} = 0,\\
&\Gamma_{01A} = \tfrac{1}{2}k'_A - \tfrac{1}{2}h_{,A}, \qquad \Gamma_{110} = h', \qquad \Gamma_{1A0} = \tfrac{1}{2}h_{,A} + \tfrac{1}{2}k'_A,\\
&\Gamma_{1AB} = \tfrac{1}{2}g'_{AB},
\end{aligned} \tag{A1}$$

$$\begin{aligned}
&\Gamma^0_{00} = \dot{h}/h - mm'/h, \qquad \Gamma^0_{\alpha 1} = 0, \qquad \Gamma^0_{0A} = (h_{,A} - k'_A)/2h,\\
&\Gamma^0_{AB} = -(1/2h)g'_{AB}, \qquad \Gamma^A_{11} = 0, \qquad \Gamma^A_{1B} = \tfrac{1}{2}g^{AC}g'_{BC},\\
&\Gamma^A_{10} = \tfrac{1}{2}g^{AB}(k'_B - h_{,B}),\\
&\Gamma^A_{00} = k_B g^{AB}[-(\dot{h}/h) + mm'/h] + g^{AB}(\dot{k}_B - mm_{,B}),\\
&\Gamma^A_{0B} = (1/2h)k_C g^{AC}(k'_B - h_{,B}) + \tfrac{1}{2}g^{AC}(\dot{g}_{BC} + k_{C,B} - k_{B,C}),\\
&\Gamma^1_{11} = h'/h,\\
&\Gamma^1_{10} = (1/2h)[2mm' - g^{AB}k'_A k_B - g^{AB}k_A h_{,B}],\\
&\Gamma^A_{BC} = \{^{\,A}_{BC}\} + (k_D g^{AD}/2h)g'_{BC},\\
&\Gamma^1_{A1} = (1/2h)(h_{,A} + k'_A - g'_{AB}g^{BC}k_C),\\
&\Gamma^1_{00} = (2m\dot{m}/h) - g^{AB}(\dot{k}_A k_B/h) + (g^{AB}/h)k_A mm_{,B},\\
&\Gamma^1_{0A} = (1/2h)(2mm_{,A} - k^B\dot{g}_{AB} + k^B[k_{A,B} - k_{B,A}]),\\
&\Gamma^1_{AB} = \tfrac{1}{2}h(k_{A,B} + k_{B,A} - \dot{g}_{AB} - 2k_C\{^{\,C}_{AB}\}),\\
&\Gamma^\mu_{\mu\alpha} = (h_{,\alpha}/h) + \tfrac{1}{2}g^{AB}g_{AB,\alpha}.
\end{aligned} \tag{A2}$$

The components of the Riemann tensor and Ricci tensor are defined as in Misner *et al* (1973). Since the formulae for an arbitrary space–time point are too long to be reproduced here, we give the values on the past light cone (with the simplifictions

(2.9)). Formula confined to the past cone are denoted by an asterisk.

$$2R_{ABCD}=2\tilde{R}_{ABCD}+\tfrac{1}{2}(\dot{g}_{AC}g'_{BD}+g'_{AC}\dot{g}_{BD}-g'_{AD}\dot{g}_{BC}-\dot{g}_{AD}\dot{g}_{BC}),\tag{A3*}$$

$$2R_{1A1B}=-g''_{AB}-h'g'_{AB}+\tfrac{1}{2}g'_{AC}g'_{BD}g^{CD},\tag{A4*}$$

$$2R_{1ABC}=g'_{AB|C}-g'_{AC|B}+\tfrac{1}{2}k'_Cg'_{AB}-\tfrac{1}{2}k'_Bg'_{AC},\tag{A5*}$$

$$2R_{0ABC}=\dot{g}_{AB|C}-\dot{g}_{AC|B}+\tfrac{1}{2}k'_B\dot{g}_{AC}-\tfrac{1}{2}k'_C\dot{g}_{AB},\tag{A6*}$$

$$2R_{0A1B}=-\dot{g}'_{AB}+k'_{B|A}+\tfrac{1}{2}k'_Ak'_B+\tfrac{1}{2}g^{CD}g'_{AC}\dot{g}_{BD},\tag{A7*}$$

$$2R_{0A0B}=-\ddot{g}_{AB}+\tfrac{1}{2}\dot{g}_{AC}\dot{g}_{BD}g^{CD},\tag{A8*}$$

$$2R_{01AB}=k'_{B,A}-k'_{A,B}+\tfrac{1}{2}g'_{AC}\dot{g}_{BD}g^{CD}-\tfrac{1}{2}g'_{AC}g'_{BD}g^{CD},\tag{A9*}$$

$$2R_{010A}=\dot{k}'_A+\tfrac{1}{2}\dot{g}_{AB}g^{BC}k'_C,\tag{A10*}$$

$$2R_{011A}=k''_A-h'_{,A}+h'k'_A-\tfrac{1}{2}g'_{AB}g^{BC}k'_C,\tag{A11*}$$

$$2R_{0101}=2\dot{h}'-\tfrac{1}{2}k'_Ak'_Bg^{AB}.\tag{A12*}$$

A vertical stroke ($|$) subscript denotes the two-dimensional covariant derivative with respect to the metric g_{AB}, the comma the ordinary derivative. The point denotes the derivative $(\partial/\partial u)$, the vertical stroke superscript is the derivative $(\partial/\partial v)$. Index displacements are carried out with the help of g_{AB} and g^{AB}. $\tilde{R}_{ABCD}$ is the Riemann tensor corresponding to g_{AB}. Since this metric is two dimensional, we have

$$\tilde{R}_{ABCD}=K(g_{AC}g_{BD}-g_{AD}g_{BC}),\qquad \tilde{R}_{AB}=Kg_{AB},\tag{A13}$$

with the Gaussian curvature K.

The relation between the Rieman tensor and Ricci tensor takes the explicit form

$$R_{00}=-2(k^A/h)R_{01A0}-R_{0A0B}g^{AB},\tag{A14a}$$

$$R_{0A}=(1/h)R_{010A}+R_{B0AC}g^{BC}-R_{10AB}(k^B/h)-R_{B0A1}(k^B/h),\tag{A14b}$$

$$R_{AB}=(1/h)R_{1AB0}-(k^C/h)R_{1ABC}+R_{ACBD}g^{CD}+(1/h)R_{1BA0}-(k^C/h)R_{1BAC},\tag{A14c}$$

$$R_{11}=-R_{1A1B}g^{AB},\tag{A14d}$$

$$R_{01}=(1/h)R_{0101}-(k^A/h)R_{01A1}+R_{0AB1}g^{AB},\tag{A14e}$$

$$R_{1A}=(1/h)R_{01A1}-R_{1BAC}g^{BC}-(k^B/h)R_{1B1A}.\tag{A14f}$$

From (A14) and (A10) the components of the Ricci tensor on the past cone are found as given in § 5. The Bianchi identities

$$g^{\rho\sigma}R_{\mu\rho;\sigma}-\tfrac{1}{2}g^{\rho\sigma}R_{\rho\sigma;\mu}=0\tag{A15}$$

are given by

$$\begin{aligned}R'_{00}-R'_{0B}k^B-R_{01,B}k^B+hg^{BC}R_{0B,C}+\dot{R}_{AB}k^B-\tfrac{1}{2}hg^{BC}\dot{R}_{BC}\\=h\Sigma^0R_{00}+h\Sigma^1R_{01}+h\Sigma^BR_{0B},\end{aligned}\tag{A16}$$

$$\dot{R}_{11}-R_{11,B}k^B+hg^{BC}R_{1B,C}-\tfrac{1}{2}g^{BC}R'_{BC}h=h\Sigma^0R_{10}+h\Sigma^1R_{11}+h\Sigma^BR_{1B},\tag{A17}$$

$$\begin{aligned}R'_{0A}+\dot{R}_{1A}-k^BR'_{AB}-R_{A1,B}k^B+g^{BC}R_{AB,C}h-R_{01,A}+R_{1B,A}k^B-\tfrac{1}{2}g^{BC}R_{BC,A}h\\=hR_{A0}\Sigma^0+hR_{A1}\Sigma^1+hR_{AB}\Sigma^B,\end{aligned}\tag{A18}$$

with

$$h\Sigma^0 = -\tfrac{1}{2}g^{BC}g'_{BC}, \tag{A19}$$

$$h\Sigma^1 = g^{11}(h' - \tfrac{1}{2}hg^{BC}g'_{BC}) + (1/h)(2\,mm' - 2k^Bk'_B + k^Bk^Cg'_{BC}) + g^{BC}k_{B|C} - \tfrac{1}{2}g^{BC}\dot{g}_{BC}, \tag{A20}$$

$$h\Sigma^A = g^{AB}(k'_B - h_{,B} - g'_{BC}k^C) + hg^{BC}\{{}^{A}_{BC}\} + \tfrac{1}{2}g^{BC}g'_{BC}k^A. \tag{A21}$$

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